# Quantization of closed orbits in Dirac theory by Maslov's complex germ method



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# Quantization of closed orbits in Dirac theory by Maslov's complex germ method

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**Abstract.** On the basis of Maslov's complex germ method for the Dirac operator in external electromagnetic and torsion fields the quasi-classical spectral series corresponding within the limit $\hbar \to 0$ to the electron motion along closed stable orbits has been constructed. The quai-classical energy spectrum is found from the condition of quantization of these orbits, and the quasi-classical asymptotics corresponding to the latter form a complete set of localized quantum state. The method is illustrated in all details by the electron motion in the axial fields as an example.

## 1. Introduction

It is well known that there is a variety of quantum-mechanical problems which can be solved if only one used a construction of quasi-classical asymptotics. A strict theory of quasi-classical asymptotics with real phases which generalizes, in the multidimensional case, the Einstein–Brillouin–Keller method, and which includes both the spectral and Couchy problems for $\hbar^{-1}$-(pseudo) differential operators was developed in papers [1, 2].

Maslov's method—a method of the canonic operator (with real phase)—imposes strict conditions on the classical system: an $n$-parametric family of $n$-dimensional Lagrangian tori which are invariant with respect to the phase flow generated by the corresponding Hamiltonian system is to exist, which, in fact, is equaivalent to its complete integrability.

When a case cannot be integrated a family of $n$-dimensional Lagrangian tori does not exist. Nevertheless, it is often the case that the Hamiltonian system which cannot be integrated, possesses tori with smaller dimensions than that of the configuration space. This situation is typical of classical systems possessing a certain (incomplete) set of symmetries, e.g. for a charge in an external electromagnetic field with axial symmetry.

A theory of the quasi-classical quantization of non-integrable Hamiltonian systems was developed in [3–6] for the scalar $\hbar^{-1}$-(pseudo) differential operators, and in [3, 7] for systems with matrix Hamiltonians as well as equations with an operator-valued symbol. The construction of the Maslov's complex germ underlies this theory.

The main point of this theory is the fact that the task of constructing the quasi-classical asymptotics $\Psi_E$ in spectral problems for both scalar and matrix $\hbar^{-1}$-(pseudo) differential operators ($E$ is a spectral parameter) is reduced to constructing geometri-

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cal objects in the $2n$-dimensional phase space—a family of the Lagrangian manifolds $\Lambda^k(\omega)$ with a complex germ $r^n(\omega)$ ($\omega=(\omega_1,\ldots,\omega_k)$ are parameters of this family $E\in\omega$). $\Lambda^k(\omega)$ result from solutions of the classical Hamiltonian system and their dimension is $0\leqslant k<n$. In the case of finite motions $\Lambda^k(\omega)$ are known as isotropic incompletely dimensional Lagrangian tori. The complex germ $r^n(\omega)$ is generated by a special set of $n$ linearly-independent complex solutions of the system in variations which are the linearization of the initial Hamiltonian system in a neighbourhood of $\Lambda^k(\omega)$.

It should be noted that the task of finding the family of isotropic manifolds with a complex germ is a separate and complicated problem in itself. In particular, in [4, 5] it was noted that the problem of the existence of a complex germ is equivalent to the orbital stability of the manifolds $\Lambda^k(\omega)$, and for the case $k=1$ it is solved in terms of the Floquet theory for linear Hamiltonian systems with periodic coefficients. At $k\geqslant 2$ the problem of constructing $r^n(\omega)$, which is naturally associated with the multidimensional analogue of the Floquet theory, has not yet been studied sufficiently (see [7] for details).

A discrete set $[\Lambda^k(\omega_N), r^n(\omega_N)]$ which generates a spectral series ($\Psi_{E_N}$, $E_N(\hbar)$, $\hbar\to 0$) is chosen from the family of isotropic tori with the complex germ $[\Lambda^k(\omega), r^n(\omega)]$ according to the quantization condition of the Bohr–Somerfeld type. Here, $N$ is a set of quantum numbers, and $E_N(\hbar)$ are eigenvalues corresponding to the quasi-classical eigenfunctions $\Psi_{E_N}$. Under these conditions it is important to take into account that in terms of the quasi-classical approach to the spectral quantum problems one deals with construction of only single spectral series in this or that domain of the energy spectrum $E'\leqslant E_N(\hbar)\leqslant E''$. The classification of the spectral series is based on that of motions of the corresponding non-integrable classical system with respect to the incompletely dimensional Lagrangian tori $\Lambda^k(\omega)$, and the numbering of the energy levels $E_N$, on the quantization conditions for these motions.

The quasi-classical asymptotics $\Psi_{E_N}$ obtained are well approximated almost everywhere by the functions of the form $\Psi_{E_N}\approx e^{i\hbar/S}\varphi$, where, unlike the usual real WKB method, the phase $S$ is complex and $\mathrm{Im}\, S\geqslant 0$. In view of this the functions $\Psi_{E_N}$ possess the following important property: within the limit $\hbar\to 0$ they are localized in a small (of the order of $\hbar^{1/2}$) neighbourhood of the 'light' domain where $\mathrm{Im}\, S=0$. This domain is the projection of a family of phase trajectories which form the tori $\Lambda^k(\omega_N)$ onto the configuration space of the classical system. Hereafter, the wavefunctions $\Psi_{E_N}$ will be referred to as the stationary trajectory-coherent states (TCS) and the corresponding approximation in the spectral problems of quantum mechanics—the stationary trajectory-coherent approximation (the TC-approximation).

The main aim of this work is to give a consistent description of construction of the quasi-classical spectral series of the Dirac operator by the complex germ method for the case when the corresponding relativistic classical system permits a family of one-dimensional ($k=1$) invariant isotropic manifolds $\Lambda^1(E)$—closed phase curves.

## 2. Complex germ method for closed orbits

### 2.1. *Families of closed phase curves with the complex germ*

Let $\lambda^{(+)}(p,q)$ be the classical Hamiltonian function defined on the $2n$-dimensional phase space $\mathbb{R}^n_p\times\mathbb{R}^n_q$, $q=(q^a)$, $p=(p_b)$, $a,b=\overline{1,n}$. Compare this with a one-parametric in $E_0$ family of the Hamiltonian functions

$$H(p, q, E_0) = \lambda^{(+)}(p, q) - E_0 \qquad E' \leqslant E_0 \leqslant E''. \tag{1}$$

Suppose that the Hamiltonian system

$$\frac{\mathrm{d}p}{\mathrm{d}\tau} = \dot{p} = -H_q(p, q, E_0) \qquad \frac{\mathrm{d}q}{\mathrm{d}\tau} = \dot{q} = H_p(p, q, E_0) \tag{2}$$

permits a smooth family of closed phase curves

$$\Lambda^1(E_0) = \{p = P(\tau, E_0), q = Q(\tau, E_0), E' \leqslant E_0 \leqslant E''\} \tag{3}$$

which satisfy the condition $|\dot{Q}| \neq 0$. The period in $\tau$ of the trajectory $\Lambda^1(E_0)$ will be denoted by $T(E_0)$.

The Hamiltonian system (2) is linearized in a neighbourhood of phase curves (3). As a result, one obtains the system in variations

$$\frac{\mathrm{d}a}{\mathrm{d}\tau}(\tau, E_0) = H_{\mathrm{VAR}}(\tau, E_0) a(\tau, E_0) \tag{4}$$

where $a(\tau, E_0) = (W(\tau, E_0), Z(\tau, E_0))^{\mathrm{T}}$ is, in a general case, the complex $2n$-dimensional vector, and $H_{\mathrm{VAR}}(\tau, E_0)$ is the $2n \times 2n$-matrix of the form

$$H_{\mathrm{VAR}}(\tau, E_0) = \begin{pmatrix} -H_{qp} & -H_{qq} \\ H_{pp} & H_{pq} \end{pmatrix} (\tau, E_0) \tag{5}$$

where $n \times n$-matrices

$$H_{qq} = \left(\frac{\partial^2 H}{\partial q^a \partial q^b}\right) \qquad H_{qp} = \left(\frac{\partial^2 H}{\partial q^a \partial p_b}\right) \text{ etc.}$$

were calculated for the points of the phase trajectory $\Lambda^1(E_0)$. Equation (4) is a linear Hamiltonian system with periodic coefficients, and one can use the general Floquet theory for such systems.

It should be reminded that the solution $a(\tau, E_0)$ of (4) is called the Floquet solution, if there exists a constant $\lambda$, the Floquet multiplicator, such that

$$a(\tau + T(E_0), E_0) = \lambda a(\tau, E_0) \qquad -\infty < \tau < \infty. \tag{6}$$

It is the multiplicators that are eigenvalues of the monodromi matrix of (4). The numbers $\omega$ determined from the condition $\lambda = \mathrm{e}^{\mathrm{i}\omega T(E_0)}$ are called the characteristic Floquet indices. One of the obvious Floquet solutions of (4) is a real solution

$$a_0(\tau, E_0) = (\dot{P}(\tau, E_0), \dot{Q}(\tau, E_0))^{\mathrm{T}} \tag{7}$$

with the multiplicator $\lambda_0 = 1$.

Now, suppose that system (4) permits a set of $n-1$ complex Floquet solutions $a_k(\tau, E_0) = (W_k(\tau, E_0), Z_k(\tau, E_0))^{\mathrm{T}}$, $k = \overline{1, n-1}$ which are linearly independent of the solution $a_0(\tau, E_0)$ and which satisfy the conditions

$$\{a_i, a_j\} = 0 \qquad \{a_l, \overset{*}{a}_k\} = 2i\delta_{lk} \qquad i, j = \overline{0, n-1} \qquad l, k = \overline{1, n-1}$$
$$a_k(\tau + T(E_0)) = \exp(\mathrm{i}\omega_k(E_0) T(E_0)) a_k(\tau) \qquad \mathrm{Im}\, \omega_k = 0 \tag{8}$$

where the symbol $*$ denotes complex conjugation and the braces $\{.\,,.\}$ imply the anti-symmetric scalar product. Then the complex $n$-dimensional plane in $\mathbb{C}^{2n}$ enveloped by the vectors $a_j(\tau, E_0)$ is called a complex germ in the point $(P(\tau, E_0), Q(\tau, E_0))$ on the phase curve $\Lambda^1(E_0)$ and is denoted as $r^n(\tau)$. A family of planes $\{r^n(\tau), 0\leqslant\tau\leqslant T(E_0)\}$ is the complex germ $r^n(E_0)$ on $\Lambda^1(E_0)$. As was pointed out in the introduction, the geometric object $[\Lambda^1(E_0), r^n(E_0)]$ is a central construction responsible for the quasi-classical results of the spectral quantum problems corresponding to the motion of the classical non-integrable system along the incompletely dimensional isotropic tori.

Hereafter, it will be supposed that the above family of the $T(E_0)$-periodic Floquet solutions (8) has been constucted. In a particular, but important for applications, case when the family of the Hamiltonian functions (1) permits the cyclic variable $\varphi(\mathrm{mod}\, 2\pi)$, the constuction of the complex germ is described in Appendix 1.

### 2.2 *Construction of the 'occupation numbers representation'*

Let $l_{E_0}=\{q^a=Q^a(\tau, E_0)\}$ be a projection of the closed curve $\Lambda^1(E_0)$ onto the configuration space $\mathbb{R}^n_q$. Consider the equation

$$\eta_{ab}(Q(\tau))\dot{Q}^a(\tau, E_0)(q^b-Q^b(\tau, E_0))=0 \tag{9}$$

where $\eta_{ab}(Q(\tau))=\eta_{ab}(q)|_{q=Q(\tau)}$, and $\eta_{ab}(q)$ is the metric of the configuration space $\mathbb{R}^n_q$. Due to the condition $|\dot{Q}|\neq 0$ in a small neighbourhood $U_\delta(Q(\tau, E_0))$ of each point $Q(\tau, E_0)$ of the curve $l_{E_0}$ equation (9) can be solved with respect to the parameter $\tau$ and, thus, it gives a parametrized-in-$\tau$ family of hyperplanes $\tau=\tau(q, E_0)$ such that

$$\dot{Q}^a\tau, a=1+O(|q-Q(\tau)|). \tag{10}$$

Henceforth, expressions like $A(\tau)$ should be understood as taken at $\tau=\tau(q, E_0)$. From the vectors $W_j(\tau, E_0)$ and $Z_j(\tau, E_0)$, which form the Floquet solutions (8) $a_j(\tau, E_0)$, $j=\overline{0, n-1}$, we shall make up square matrices $B(\tau, E_0)$ and $C(\tau, E_0)$ of order $n$

$$B=(\dot{P}, W_1, \ldots, W_{n-1}) \qquad C=(\dot{Q}, Z_1, \ldots, Z_{n-1}). \tag{11}$$

The matrix $C(\tau, E_0)$ is non-singular and, in this way, we find the symmetric matrix $G=BC^{-1}$ with the positively defined imaginary part

$$\mathrm{Im}\, G>0. \tag{12}$$

We shall introduce a complex phase $S(q, E_0)=[\tilde{S}(\tau, q, E_0)]|_{\tau=\tau(q)}$ in the neighbourhood $U_\delta(l_{E_0})$, where

$$\tilde{S}(\tau, q, E_0)=\int_0^\tau \langle P(t, E_0), \dot{Q}(t, E_0)\rangle\, \mathrm{d}t+(E-E_0)\tau$$
$$+\langle P(\tau, E_0, \Delta q\rangle+\tfrac{1}{2}\langle\Delta q, G(\tau, E_0)\Delta q\rangle. \tag{13}$$

Hereafter $\langle .\,, .\rangle$ implies a Euclidean scalar vector product, $\Delta q=q-Q(\tau, E_0)$, and $E=E_0+\hbar E_1+O(\hbar^2)$.

We shall introduce a class of functions of the form

$$Y^S_\hbar=\left\{N_0(\hbar)\exp\left(\frac{\mathrm{i}}{\hbar}S(q, E_0)\right)\sum_{|\kappa|}^{N} c_\kappa(\tau)(\Delta q/\sqrt{\hbar})^\kappa, N=0, 1, \ldots\right\} \tag{14}$$

where $\kappa$ is a multi-index: $\kappa=(\kappa_1, \kappa_2, \ldots, \kappa_n)$, $|\kappa|=\kappa_1+\kappa_2+\cdots+\kappa_n$, $(\Delta q/\sqrt{\hbar})^\kappa=\Pi^n_{a=1}(\Delta q^a/\sqrt{\hbar})^{\kappa_a}$, $c_\kappa(\tau)$: $\mathbb{R}^n_q\to\mathbb{C}^1$, $S(q, E_0)$ is the phase defined above, and $N_0(\hbar)$ is the normalization factor (with respect to $L_2$-norm). Let the operator $\hat{F}$: $Y^S_\hbar\to Y^S_\hbar$ be denoted through $\hat{O}(\hbar^{\alpha})$, $\alpha>0$. For this operator an asymptotic estimation $\|\hat{F}\varphi\|_{L_2(\mathbb{R}_q^n)}=O(\hbar^{\alpha})$, $\forall\varphi\in Y_{\hbar}^{S}$ in the limit at $\hbar\to 0$ is fulfilled. In particular, for the operators

$$\Delta q=q-Q(\tau,E_0)\qquad \Delta\hat{p}=\hat{p}-P(\tau,E_0)\tag{15}$$

where $\hat{p}=-i\hbar\partial_q|_{\tau=\text{const}}$, we shall have

$$\Delta q=\hat{O}(\hbar^{1/2})\qquad \Delta\hat{p}=\hat{O}(\hbar^{1/2}).\tag{16}$$

The creation and annihilation operators are associated with the vectors $a_j(\tau,E_0)$, $\overset{*}{a}_j(\tau,E_0)$ by the rule

$$\begin{aligned}
&\hat{a}_0=\langle\dot{Q}(\tau),\Delta\hat{p}\rangle-\langle\dot{P}(\tau),\Delta q\rangle\\
&\hat{a}_l=\frac{1}{(2\hbar)^{1/2}}(\langle Z_l(\tau),\Delta\hat{p}\rangle-\langle W_l(\tau),\Delta q\rangle)\\
&\hat{a}_l^{+}=\frac{1}{(2\hbar)^{1/2}}(\langle\overset{*}{Z}_l(\tau),\Delta\hat{p}\rangle-\langle\overset{*}{W}_l(\tau),\Delta q\rangle)\qquad l=\overline{1,n-1}
\end{aligned}\tag{17}$$

for which the usual Bose commutation rules hold

$$\begin{aligned}
&[\hat{a}_0,\hat{a}_l]=[\hat{a}_0,\hat{a}_l^{+}]=[\hat{a}_k,\hat{a}_l]=[\hat{a}_k^{+},\hat{a}_l^{+}]=0\\
&[\hat{a}_k,\hat{a}_l^{+}]=\delta_{kl}.
\end{aligned}\tag{18}$$

We shall introduce a 'vacuum' state

$$|0,\tau\rangle=N_0(\hbar)J^{-1/2}\exp\left(\frac{i}{\hbar}S(q,E_0)\right)\tag{19}$$

where $J(\tau,E_0)=\det C(\tau,E_0)$, and by using the creation operators $\hat{a}_l^{+}$ we shall construct the 'occupation numbers representation', i.e., a set of functions of the form

$$|\nu,\tau\rangle=\prod_{l=1}^{n-1}\frac{1}{(\nu_k!)^{1/2}}(\hat{a}_l^{+})^{\nu_l}|0,\tau\rangle.\tag{20}$$

It is not difficult to see that functions (20) belong to $Y_{\hbar}^{S}$. We shall compare the classical Hamiltonian function $\lambda^{(+)}(p,q)$ (1) with the Weyl-ordered quadratic operator

$$\begin{aligned}
\hat{\lambda}_0^{(+)}=E_0+\hat{a}_0+\tfrac{1}{2}\{&\langle\Delta q,\lambda_{qq}^{(+)}(\tau)\Delta q\rangle+\langle\Delta q,\lambda_{qp}^{(+)}(\tau)\Delta\hat{p}\rangle+\langle\Delta\hat{p},\lambda_{pq}^{(+)}(\tau)\Delta q\rangle\\
&+\langle\Delta\hat{p},\lambda_{pp}^{(+)}(\tau)\Delta\hat{p}\rangle\}.
\end{aligned}\tag{21}$$

Here

$$\lambda_{qq}^{(+)}(\tau)=\lambda_{qq}^{(+)}(p,q)\bigg|_{\substack{p=P(\tau,E_0)\\ q=Q(\tau,E_0)}}$$

etc. Then the asymptotic estimation $[-i\hbar\partial_\tau+(E_0-E)+\hat{a}_0]=O(\hbar)$ holds on the set $Y_{\hbar}^{S}$. As a result of equations (18) and (19) it is not difficult to show that functions (20) are the 'vacuum' ones for the operator $\hat{a}_0$

$$\hat{a}_0|\nu,\tau\rangle=0\tag{22}$$

and satisfy the Schrödinger equation with the square-law Hamiltonian (21)

$$(-i\hbar\partial_\tau+\hat{\lambda}_0^{(+)}-E)|\nu,\tau\rangle=0.\tag{23}$$

## 3. Quasi-classical spectral series of the Dirac operator corresponding to the family of closed phase trajectories $\Lambda^1(E_0)$

In this section asymptotic spectral series corresponding at $\hbar\to 0$ to motion of a classical particle along closed orbits of the configuration space are constructed for the Dirac operator in external electromagnetic and torsion fields.

### 3.1. Statement of the problem

Let the notation used hereafter be specified. The Cartesian coordinates of the Minkowski space with the signature $(+,-,-,-)$ will be denoted as $x^i=(ct,x,y,z)=(x^0,x^{\bar a})$, $\bar a,\bar b,\bar c=1,2,3$; $i,j,k=0,1,2,3$ and the curvilinear coordinates $q^\mu=(q^0,q^a)$, $a,b,c=1,2,3$; $\alpha,\beta,\mu,\nu=0,1,2,3$, respectively. We shall restrict ourselves to the curvilinear coordinate systems with a stationary metric $\mathrm{d}s^2=c^2\,\mathrm{d}t^2-\eta_{ab}(q)\mathrm{d}q^a\mathrm{d}q^b$, where

$$\eta_{ab}=\frac{\partial x^{\bar a}}{\partial q^a}\delta_{\bar a\bar b}\frac{\partial x^{\bar b}}{\partial q^b}.$$

If one introduces three vectors $\mathbf{e}_a$ with the components $e_a^{\bar a}=\partial x^{\bar a}/\partial q^a$, the metric tensor $\eta_{ab}$ can be represented in the form $\eta_{ab}=\langle\mathbf{e}_a,\mathbf{e}_b\rangle$.

The Dirac equation in the arbitrary curvilinear coordinate system $q^\mu$ in the Minkowski space will be written as

$$\left(\gamma^\mu\hat P_\mu-mc-\frac{3\mathrm{i}\hbar}{2}\overset{*}{\gamma}{}^\mu S_\mu\right)\Psi=0 \tag{24}$$

where

$$\hat P_\mu=\mathrm{i}\hbar\frac{\partial}{\partial q^\mu}-\frac{e}{c}A_\mu$$

$e=-e_0$ is the electron charge, $A_\mu$ are potentials of the external electromagnetic field, and $S_\mu$ is the torsion pseudo-vector. The Dirac matrices $\gamma^\mu$, $\overset{*}{\gamma}{}^\mu$ are defined by the conditions $\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}$, $\overset{*}{\gamma}{}^\mu=\mathrm{i}\gamma^5\gamma^\nu$, $\gamma^5=-(\mathrm{i}/4!)e_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta$ and can be taken in the form $\gamma^0=\rho_3$, $\gamma^a=\eta^{ab}e_b^{\bar a}\gamma^{\bar a}$, $\gamma^5=-\rho_1$, where $\boldsymbol\gamma=(\gamma^{\bar a})=\rho_3\boldsymbol\alpha$, $\boldsymbol\alpha=\rho_1\boldsymbol\Sigma$. Here $\rho_1$, $\rho_3$, $\boldsymbol\Sigma$ are the Dirac plane matrices in a standard representation.

In our case the Dirac equation (24) will take the form

$$(-\mathrm{i}\hbar\partial_\tau+\hat H_{\mathrm{D}})\Psi=0 \tag{25}$$

where the Weyl-ordered operator $\hat H_{\mathrm{D}}$ may be represented as follows

$$\hat H_{\mathrm{D}}=\hat H_0+\hbar\hat H_1 \tag{26}$$

where

$$\hat H_0=-\frac{c}{2}\alpha^{\bar a}(e_{\bar a}^a\hat P_a+\hat P_a e_{\bar a}^a)+\rho_3 mc^2+eA_0 \tag{27}$$

$$\hat H_1=\frac{\mathrm{i}c}{2}\alpha^{\bar a}e_{\bar a,a}^a+\frac{3c}{2}(-\rho_1 S_0+\boldsymbol\Sigma\mathbf{S}). \tag{28}$$

Here $\alpha^{\tilde{a}}$ are the components of the vector-matrix $\boldsymbol{\alpha}$, and $S_i=(S_0, -\boldsymbol{S})$ are the Cartesian components of the torsion pseudo-vector.

Let the spectral problem

$$(\hat{H}_{\mathrm{D}}-\mathrm{E})\Psi_E=0 \tag{29}$$

be considered for the Dirac operator $\hat{H}_{\mathrm{D}}$ (26). In (29) $E$ is the spectral parameter. On the set of equation (29) solutions one can introduce the inner product

$$\langle\Psi_{E'}|\Psi_E\rangle_{\mathrm{D}}=\int \mathrm{d}^3q\sqrt{g}\overset{+}{\Psi}_{E'}\Psi_E \qquad g=\det(\eta_{ab}). \tag{30}$$

One may denote the closed phase trajectory of a classical electron with the energy $E_0$ through $\Lambda^1(E_0)=\{p=P(\tau, E_0),\ q=Q(\tau, E_0)\}$ where the functions $P(\tau, E_0)$, $Q(\tau, E_0)$ are the $T(E_0)$-periodic solutions of the Hamiltonian system (1) with the relativistic Hamiltonian $\lambda^{(+)}(p,q)=eA_0+[c^2P^2+m^2c^4]^{1/2}$, $\boldsymbol{P}=(P_{\tilde{a}})$, $P_{\tilde{a}}=e^a_{\tilde{a}}(p_a+(e/c)A_a)$.

It is assumed that the sequence of the numbers $E_N=E_N(\hbar)$ (where $N$ is a set of the corresponding quantum numbers) and that of the functions $\Psi_{E_N}(q,\hbar)$ make a quasi-classical spectral series of the Dirac operator $\hat{H}_{\mathrm{D}}$ corresponding in the limit $\hbar\to 0$ to the family of closed phase curves $\Lambda^1(E_0)$, if the following conditions hold:

(1) $$\lim_{\hbar\to 0} E_N(\hbar)=E_0 \qquad E'\leqslant E_0\leqslant E'' \tag{31}$$

(which implies a correspondence of the quasi-classical set of energy levels $E_N(\hbar)$ to classical motion with energy $E_0$).

(2) $$(\hat{H}_{\mathrm{D}}-E_N(\hbar))\Psi_{E_N(\hbar)}(q,\hbar)=O(\hbar^{3/2}). \tag{32}$$

(3) The functions $\Psi_{E_N(\hbar)}(q,\hbar)$ at $\hbar\to 0$ are localized in a small tube-like neighbourhood $U_\delta(l_{E_0})$ with the diameter $\delta\approx\hbar^{1/2}$ of the closed curve $l_{E_0}=\{q=Q(\tau, E_0)\}$.

(4) The functions $\Psi_{E_N(\hbar)}(q,\hbar)$ form at $\hbar\to 0$ the orthonormal set of states with an accuracy of $O(\hbar^{1/2})$

$$\langle\Psi_{E_{N'}}|\Psi_{E_N}\rangle_{\mathrm{D}}=\delta_{N'N}+O(\hbar^{1/2}). \tag{33}$$

Taking into account these properties we call the asymptotic eigenfunctions $\Psi_{E_N}$ the stationary trajectory-coherent states (TCS), and the corresponding approximation of (1)–(4) for equation (29) a stationary trajectory-coherent approximation. The dynamic TCS of the Dirac equation were constructed earlier in papers [10].

### 3.2. *Stationary TCS of the Dirac operator* $\hat{H}_{\mathrm{D}}$

Consider the problem of constructing explicitly a set of approximated solutions $\Psi_{E_N}$ of the Dirac equation (29) which satisfy condition (32).

The main symbol-matrix $H_0(p,q)$ of the form

$$H_0(p,q)=c\boldsymbol{\alpha P}+\rho_3 mc^2+eA_0 \tag{34}$$

corresponds to the Dirac operator $\hat{H}_{\mathrm{D}}$ (26). Let the spectral properties of matrix (34) be considered. The equation

$$H_0 f^{(\pm)}=\lambda^{(\pm)}f^{(\pm)} \tag{35}$$

possesses the two two-fold degenerate eigenvalues

$$\lambda^{(\pm)}(p, q) = eA_0 \pm \varepsilon \qquad \varepsilon = (c^2\boldsymbol{P}^2 + m^2c^4)^{1/2}. \tag{36}$$

The eigenvectors $f_j^{(\pm)}(p, q)$, $j = 1, 2$ corresponding to them are combined into two $4 \times 2$-matrices

$$\begin{aligned} \Pi_+(p, q) &= \frac{1}{[2\varepsilon(\varepsilon + mc^2)]^{1/2}} \begin{pmatrix} \varepsilon + mc^2 \\ c\boldsymbol{\sigma}\boldsymbol{P} \end{pmatrix} \\ \Pi_-(p, q) &= \frac{1}{[2\varepsilon(\varepsilon + mc^2)]^{1/2}} \begin{pmatrix} c\boldsymbol{\sigma}\boldsymbol{P} \\ -(\varepsilon + mc^2) \end{pmatrix} \end{aligned} \tag{37}$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. It is assumed that the Hamiltonian function $\lambda^{(\pm)}(p, q)$ permits a family of closed phase trajectories with the complex germ: $[\Lambda^1(E_0), r^3(E_0)]$. Then it is possible to use the results of section 2. All the denotations are preserved.

A number of identities will be required below, to which the matrices

$$\Pi_\pm(\tau) = \Pi_\pm(p, q)\Big|_{\substack{p = P(\tau) \\ q = Q(\tau)}}$$

hold true, and, according to (37) one has

$$\begin{aligned} \Pi_+(\tau) &= \frac{1}{[2(1 + \gamma^{-1})]^{1/2}} \begin{pmatrix} 1 + \gamma^{-1} \\ (1/c)\boldsymbol{\sigma}\dot{\boldsymbol{q}} \end{pmatrix} \\ \Pi_-(\tau) &= \frac{1}{[2(1 + \gamma^{-1})]^{1/2}} \begin{pmatrix} (1/c)\boldsymbol{\sigma}\dot{\boldsymbol{q}} \\ -(1 + \gamma^{-1}) \end{pmatrix} \end{aligned} \tag{38}$$

where $\gamma = \varepsilon/mc^2$, $\dot{\boldsymbol{q}} = \boldsymbol{e}_a(\tau)\dot{Q}^a$ and $\boldsymbol{e}_a(\tau) = \boldsymbol{e}_a(q)|_{q=Q(\tau)}$. The matrices $\Pi_\pm(\tau)$ satisfy the orthogonality and completeness conditions

$$\overset{+}{\Pi}_{\xi'}\Pi_\xi = \delta_{\xi'\xi} \qquad \sum_\xi \Pi_\xi \overset{+}{\Pi}_\xi = 1 \qquad \xi = \pm 1. \tag{39}$$

Also, let $\boldsymbol{a}$ be an arbitrary vector, then

(1)
$$\langle \boldsymbol{\alpha}, \boldsymbol{a}\rangle \Pi_\pm = \pm \frac{1}{c} \Pi_\pm \langle \dot{\boldsymbol{q}}, \boldsymbol{a}\rangle + \Pi_\mp \langle \boldsymbol{d}_1, \boldsymbol{a}\rangle, \tag{40}$$

where

$$\boldsymbol{d}_1 = \langle \boldsymbol{\sigma}, \dot{\boldsymbol{q}}\rangle \frac{\dot{\boldsymbol{q}}}{c^2(1 + \gamma^{-1})} - _{\mathrm{i}}\boldsymbol{\sigma}.$$

(2)
$$\langle \boldsymbol{\Sigma}, \boldsymbol{a}\rangle \Pi_\pm = \Pi_\pm \langle \boldsymbol{d}_2, \boldsymbol{a}\rangle \pm \frac{}{c} \Pi_\mp \langle \boldsymbol{\sigma}, \dot{\boldsymbol{q}} \times \boldsymbol{a}\rangle, \tag{41}$$

where

$$\boldsymbol{d}_2 = \langle \boldsymbol{\sigma}, \dot{\boldsymbol{q}}\rangle \frac{\dot{\boldsymbol{q}}}{c^2(1 + \gamma^{-1})} + \gamma^{-1}\boldsymbol{\sigma}.$$

(3)
$$\rho_3 \Pi_\pm = \pm \gamma^{-1} \Pi_\pm + \frac{1}{c} \Pi_\mp \langle \boldsymbol{\sigma}, \dot{\boldsymbol{q}}\rangle. \tag{42}$$

(4)
$$\rho_1 \Pi_\pm = \pm \frac{1}{c} \Pi_\pm \langle \boldsymbol{\sigma}, \dot{\boldsymbol{q}}\rangle - \gamma^{-1} \Pi_\mp. \tag{43}$$

(5)
$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Pi_{\pm}=\Pi_{\pm}d_3\mp\Pi_{\mp}d_4 \tag{44}$$

where

$$d_3=\frac{\mathrm{i}}{2}\frac{\langle\boldsymbol{\sigma},\dot{\boldsymbol{q}}\times\boldsymbol{q}\rangle}{c^2(1+\gamma^{-1})}\qquad d_4=\frac{1}{2c}\left\langle\boldsymbol{\sigma},\left(\dot{\boldsymbol{q}}\frac{\gamma\langle\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}\rangle}{c^2(1+\gamma^{-1})}+\ddot{\boldsymbol{q}}\right)\right\rangle$$

and

$$\ddot{\boldsymbol{q}}=\frac{\mathrm{d}}{\mathrm{d}\tau}\dot{\boldsymbol{q}}.$$

The asymptotic eigenfunctions $\Psi_E(q,\hbar)$ of the Dirac operator (26) are sought for as

$$\Psi_E(q,\hbar)=\Psi_E(q,\tau(q),\hbar). \tag{45}$$

The asymptotic eigenvalues $E$ corresponding to them and satisfying condition (31) are found as the following expansion over $\hbar$: $E=E_0+\hbar E_1+O(\hbar^2)$. In this case (32) results in the following conditions

$$\hat{a}_0\Psi_E(q,\hbar)=O(\hbar) \tag{46}$$

$$\left(-\mathrm{i}\hbar c\langle\boldsymbol{\alpha},\nabla\tau\rangle\frac{\partial}{\partial\tau}+\hat{H}_{\mathrm{D}}|_{\tau=\mathrm{const}}-E\right)\Psi_E(q,\hbar)=O(\hbar^{3/2}) \tag{47}$$

where

$$\nabla=e^a\frac{\partial}{\partial q^a}.$$

Let us represent $\Psi_E$ as a linear combination

$$\Psi_E(q,\tau)=[\Pi_+(\tau)u_E^{(+)}(q,\hbar)+\Pi_-(\tau)u_E^{(-)}(q,\hbar)]. \tag{48}$$

It is assumed that the two-component spinors $u_E^{(\pm)}$ can be represented as $u_E^{(+)}(q,\hbar)=g^{-1/4}(\tau)v(\tau)\varphi(q,\hbar)$, where $\varphi(q,\hbar)\in Y_\hbar^S$ and the spinors $v(\tau)$ are liable to determination. Let the left-hand side of (47) be expanded in the Taylor power series over the operator $\Delta q$ and $\Delta\hat{p}$ to the second order included, and the asymptotic estimations (16) be used. Then, substituting (48) into (47) and making use of equations (40)–(44), one obtains

$$\begin{aligned}\Pi_+[\{\hat{F}_1+\langle\dot{\boldsymbol{q}},\Delta\boldsymbol{P}\rangle+e\Delta A_0+\lambda^{(+)}(\tau)-\mathrm{i}\hbar(d_3+d_4d_5)+\hbar\overset{+}{\Pi}_+H_1\Pi_+-E\}u_E^{(+)}\\-\{d_5\hat{F}_1-c\langle\boldsymbol{d}_1,\Delta\boldsymbol{P}\rangle+\mathrm{i}\hbar(d_4-d_3d_5)-\hbar\overset{+}{\Pi}_+H_1\Pi_-\}u_E^{(-)}\\+\Pi_-[\{-\hat{F}_1-\langle\dot{\boldsymbol{q}},\Delta\boldsymbol{P}\rangle+e\Delta A_0+\lambda^{(-)}(\tau)+\mathrm{i}\hbar(d_3+d_4d_5)\\+\hbar\overset{+}{\Pi}_-H_1\Pi_--E\}u_E^{(-)}\\-\{d_5\hat{F}_1-c\langle\boldsymbol{d}_1,\Delta\boldsymbol{P}\rangle+\mathrm{i}\hbar(d_4-d_3d_5)-\hbar\overset{+}{\Pi}_-H_1\Pi_+\}u_E^{(+)}]=0(\hbar^{3/2}).\end{aligned} \tag{49}$$

In (49) through $\hat{F}_1$ and $d_5$ we denote the following expressions: $\hat{F}_1=(-\mathrm{i}\hbar\partial_\tau+\hat{\lambda}_0^{(+)}-E)-\hat{a}_0-\frac{1}{2}\hat{\delta}^2\lambda^{(+)}+\hbar E_1$, $d_5=\langle\boldsymbol{\sigma},\dot{\boldsymbol{q}}\rangle/[c\gamma(1-\gamma^{-2})]$ and $\Delta$ is the square-law operator of the type $\Delta=\hat{\delta}^1+\frac{1}{2}\hat{\delta}^2$. The expression $\hat{\delta}^kA(\tau)$ implies the $k$th term in the Taylor power expansion over the $\Delta q$ and $\Delta\hat{p}$ operators of the Weyl-ordered operator $\hat{A}=A(\hat{p},q)$ with the symbol $A(p,q)$:

$$\hat{\delta}^k A(\tau)=\left\{\left(\left\langle\Delta\hat{p},\frac{\partial}{\partial\bar{p}}\right\rangle+\left\langle\Delta q,\frac{\partial}{\partial\bar{q}}\right\rangle\right)^k A(\bar{p},\bar{q})\right\}\Bigg|_{\substack{\bar{p}=P(\tau)\\ \bar{q}=Q(\tau)}}. \tag{50}$$

Let the spinors $u_E^{(\pm)}$ in (49) be expanded over $\hbar^{1/2}$

$$u_E^{(\pm)}=u_E^{0(\pm)}+\hbar^{1/2}u_E^{1(\pm)}+\hbar u_E^{2(\pm)}+O(\hbar^{3/2}) \tag{51}$$

and the terms to the same powers of $\hbar$ be gathered together, where a chain of the following conditions is obtained

$$u_E^{0(-)}=0 \qquad u_E^{1(-)}=\frac{1}{2\varepsilon}\hat{\tilde{Q}}_1 u_E^{0(+)} \qquad u_E^{2(-)}=\frac{1}{2\varepsilon}(\hat{\tilde{Q}}_2 u_E^{0(+)}+\hat{\tilde{Q}}_1 u_E^{1(+)}) \tag{52}$$

where the $\hat{\tilde{Q}}_1$ and $\hat{\tilde{Q}}_2$ operators have the form

$$\hbar^{1/2}\hat{\tilde{Q}}_1=d_5\hat{a}_0+\hbar^{1/2}\hat{Q}_1 \qquad \hbar^{1/2}\hat{Q}_1=c\langle \boldsymbol{d}_1,\hat{\delta}^1\boldsymbol{P}\rangle \tag{53}$$

$$\hbar\tilde{Q}_2=\hbar^{1/2}\frac{e}{\varepsilon}(\hat{\delta}^1A_0)\hat{\tilde{Q}}_1-d_5(\hat{F}_1+\hat{a}_0)+\tfrac{1}{2}c\langle \boldsymbol{d}_1,\hat{\delta}^2\boldsymbol{P}\rangle-\mathrm{i}\hbar(d_4-d_3d_5)+\hbar\overset{+}{\Pi}_-H_1\Pi_+. \tag{54}$$

Taking into account equations (52)–(54) a similar procedure for the expression with the symbol $\Pi_+$ in (49) results in the equation for the two-components spinor $u_E^{0(+)}$ which, after further simplifications, is reduced to

$$\Bigg[-\mathrm{i}\hbar\partial_\tau+\hat{\lambda}_0^{(+)}-E-\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}\tau}\ln(\sqrt{g}(\tau))^{1/4}+\hbar\langle\boldsymbol{\sigma},\mathfrak{B}(\tau)\rangle$$
$$+\frac{\ddot{q}\hat{a}_0^2}{2\varepsilon c^2\gamma^2(1-\gamma^{-2})^2}+\frac{\hbar^{1/2}}{\varepsilon}d_5\hat{Q}_1\hat{a}_0\Bigg]u_E^{0(+)}=0. \tag{55}$$

Here the vector $\mathfrak{B}(\tau)$ (the 'polarization' vector) is given by

$$\mathfrak{B}(\tau)=-\frac{ec}{2\varepsilon}\left(\boldsymbol{H}(\tau)-\frac{\dot{\boldsymbol{q}}\times\boldsymbol{E}(\tau)}{c(1+\gamma^{-1})}\right)+\frac{3c}{2}\left(-\frac{\dot{\boldsymbol{q}}}{c}S_0(\tau)+\gamma^{-1}\boldsymbol{S}(\tau)+\dot{\boldsymbol{q}}\frac{\langle\dot{\boldsymbol{q}},\boldsymbol{S}(\tau)\rangle}{c^2(1+\gamma^{-1})}\right) \tag{56}$$

where $\boldsymbol{E}(\tau)$ and $\boldsymbol{H}(\tau)$ are the Cartesian components of the external electromagnetic field. If in (55) the spinor $u_E^{0(+)}$ is taken to have the following form

$$u_E^{0(+)}=(g(\tau))^{-1/4}|\nu,\tau\rangle v(\tau) \tag{57}$$

then, as it is not difficult to see, in view of (22) and (23), the variable $\tau$ gets detached and the equation takes the form of the matrix equation of the Pauli type for the two-component spinor $v(\tau)$

$$\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\tau}+\langle\boldsymbol{\sigma},\mathfrak{B}(\tau)\rangle\right)v(\tau)=0. \tag{58}$$

Hence, a remarkable conclusion follows: the problem of constructing the states $\Psi_E$ satisfying equation (47) is reduced to solution of an ordinary linear differential system (58) relative to the variable $\tau$ with a subsequent substitution of the function $\tau=\tau(q)$ into the solution $(v(\tau)\to v(\tau(q)))$.

Using (45), (48) and (52)–(54) one can get the explicit expression for the function $\Psi_E(q,\hbar)$

$$\Psi_E(q,\hbar)=\left[\Pi_+(\tau)+\frac{1}{2\varepsilon}\Pi_-(\tau)(\hbar^{1/2}\hat{\hat{Q}}_1+\hbar\hat{\hat{Q}}_2)\right]u_E^{0(+)}(q,\hbar)$$

$$+\hbar^{1/2}\left[\Pi_+(\tau)+\frac{\hbar^{1/2}}{2\varepsilon}\Pi_-(\tau)\hat{\hat{Q}}_1\right]u_E^{1(+)}(q,\hbar)+\hbar\Pi_+(\tau)u_E^{2(+)}(q,\hbar). \qquad (59)$$

It follows from (22) and (57), first, that the functions $\Psi_E$ satisfies (46) directly, and, second, that they are determined only with an accuracy to $O(\hbar^{1/2})$, since they contain the two arbitrary two-component spinors $u_E^{1(+)}$ and $u_E^{2(+)}$. They can be found from the higher orders.

Hereafter, we shall deal only with the leading term of the asymptotic (59) which, according to (45) and (57), has the form

$$\mathring{\Psi}_E(q,\hbar)=\left(\Pi_+(\tau)v(\tau)\frac{|\nu,\tau\rangle}{(g(\tau))^{1/4}}\right)\Bigg|_{\tau=\tau(q)}. \qquad (60)$$

### 3.3. *Quantization condition of closed orbits* $\Lambda^1(E_0)$

To construct a quasi-classical spectral series of the Dirac operator $\hat{H}_D$ let the functions satisfying the periodicity condition be selected from the family of functions (60)

$$\mathring{\Psi}_E(q,\tau+T(E_0))=\mathring{\Psi}_E(q,\tau). \qquad (61)$$

It can be shown that this condition leads to the Bohr–Sommerfeld type quantization of the family $\Lambda^1(E_0)$ and distinguishes a discrete set of energy levels $E_N(\hbar)$ in the domain of continuous changing of the parameter $E_0(E'\leqslant E_0\leqslant E'')$.

It follows from the results obtained in the previous section that in the stationary TC-state the interaction of the electron spin with the external field is described by equations (58) when an electron motion is taken to occur along a closed orbit $\Lambda^1(E_0)$. Since there is the condition $\mathfrak{B}(\tau+T(E_0))=\mathfrak{B}(\tau)$, equation (58) represents a linear Hamiltonian system with periodic coefficients. Like in the case of the Hamiltonian system in variation (4) it will be assumed that (58) permits a set of two lineearly independent Floquet solutions $v_\zeta$, $\zeta=\pm1$ satisfying the orthonormality and completeness condition

$$v_\zeta(\tau+T(E_0))=\exp(-\mathrm{i}\omega_\zeta^s(E_0)T(E_0))v_\zeta(\tau) \qquad \mathrm{Im}\,\omega_\zeta^s(E_0)=0 \qquad (62)$$

$$\overset{+}{v}_{\zeta'}v_\zeta=\delta_{\zeta'\zeta} \qquad \sum_\zeta v_\zeta\overset{+}{v}_\zeta=1. \qquad (63)$$

Some of the principal properties of the Floquet solutions of (58) are given in Appendix 2.

Now it is not difficult to find the conditions under which relation (61) holds. Substituting into (61) the explicit form of the function $\mathring{\Psi}_E$ (60) and making use of (8) and (62) one gets the quantization condition for the family $[\Lambda^1(E_0), r^3(E_0)]$ as follows

$$\int_0^{T(E_0)}\langle P(\tau,E_0),\dot{Q}(\tau,E_0)\rangle\mathrm{d}\tau+\hbar T(E_0)E_1$$

$$=2\pi\hbar l+\hbar T(E_0)\left(\sum_{k=1}^{2}\omega_k(E_0)(\nu_k+1/2)+\omega_\zeta^s(E_0)\right)+O(\hbar^2). \qquad (64)$$

Whence it follows, for example, that (61) is sure to be fulfilled if the conditions are satisfied

$$\int_0^{T(E_0)} \langle P(\tau, E_0), \dot{Q}(\tau, E_0)\rangle \mathrm{d}\tau = 2\pi\hbar l(\hbar) \tag{65}$$

$$E_1 = \sum_{k=1}^{2} \omega_k(E_0)(\nu_k + 1/2) + \omega_\zeta^\xi(E_0). \tag{66}$$

Here the integer sequence $l(\hbar)$ and the parameter $\hbar$ should be tied by the condition

$$\lim_{\hbar\to 0} \hbar l(\hbar) = \frac{1}{2\pi}\int_0^{T(E_0)} \langle P(\tau, E_0), \dot{Q}(\tau, E_0)\rangle \mathrm{d}\tau$$

where $E_0$ is the given value of the electron classical energy corresponding to the closed orbit $\Lambda^1(E_0)$. In this case condition (65) defines a discrete set of energy values $E_l^{(0)}(\hbar) = E_0(\hbar l(\hbar))$ in the neighbourhood of $E_0$ such that

$$\lim_{\hbar\to 0} E_l^{(0)}(\hbar) = E_0.$$

Thus, equations (65) and (66) determine the spectral sequence of energy levels up to $O(\hbar^2)$ as follows

$$E_N(\hbar) = E_{l,\nu_1,\nu_2,\zeta}(\hbar) = E_l^{(0)}(\hbar) + \hbar E^{(1)}_{l,\nu_1,\nu_2,\zeta}(\hbar) + O(\hbar^2) \tag{67}$$

where

$$E_l^{(0)}(\hbar) = E_0(\hbar l(\hbar)) \tag{68}$$

$$E^{(1)}_{l,\nu_1,\nu_2,\zeta}(\hbar) = \sum_{k=1}^{2} \omega_k(E_l^{(0)}(\hbar))(\nu_k + 1/2) + \omega_\zeta^\xi(E_l^{(0)}(\hbar)). \tag{69}$$

It can be shown (see Appendix 3) that in expanding with respect to $\hbar$, $\hbar \to 0$, the quantization conditions (65), (66) are equivalent with the accuracy to $O(\hbar^2)$ to the quantization condition of the Bohr–Sommerfeld type for the family $\Lambda^1(E)$, where $E = E_0 + \hbar E_1 + O(\hbar^2)$

$$\frac{1}{2\pi\hbar}\oint_{\Lambda^1(E)} \langle P(\tau, E), \mathrm{d}Q(\tau, E)\rangle = l(\hbar) + \frac{T(E)}{2\pi}\left(\sum_{k=1}^{2} \omega_k(E)(\nu_k + 1/2) + \omega_\zeta^\xi(E)\right). \tag{70}$$

## 4. Quasi-classical spectral series of the Dirac operator in axially symmetric electromagnetic fields

As follows from the previous sections, while defining the spectral series of the Dirac operator, the construction of the complex germ $r^n(E_0)$, i.e. a set of $n$ Floquet solutions of the system in variations satisfying condition (8) is mainly used. The conditions which the family of closed phase curves $\Lambda^1(E_0)$ is to satisfy so that the complex germ should exist, may be obtained, as has been pointed out in the introduction, in terms of the Floquet general theory and they are given in [4].

In the case when the classical Hamiltonian function permits only one cyclic angular variable (see Appendix 1), the problem of the complex germ existence is solved in a fairly simple way (see, for example, [5]). Below three examples of such systems are considered, and the quasi-classical spectral series of the Dirac operator corresponding to an electron motion along an equilibrium circle are constructed for the family $\Lambda(I)$. To simplify the description, in the first and second cases the torsion fields are taken to be equal to zero.

### *4.1. Spectral series of relativistic electron moving in axially symmetric magnetic field with weak focusing*

In cylindrical coordinates $(q^a)=(\rho, \varphi, z)$ let the electromagnetic field potentials be taken as

$$A_\rho=0 \qquad A_\varphi=\frac{b\rho^{2-q}}{2-q}\left[1+\frac{q(2-q)}{2}\frac{z^2}{\rho^2}\right] \qquad A_z=A_0=0 \tag{71}$$

where $0<q<1$ is the focusing parameter, and $b=\text{constant}$. The corresponding relativistic Hamiltonian function is as follows

$$\lambda^{(+)}(p,q)=c\left(p_\rho^2+\rho^{-2}\left(p_\varphi+\frac{e}{c}A_\varphi\right)^2+p_z^2+m^2c^2\right)^{1/2}=c\varepsilon(p,q). \tag{72}$$

Function (72) permits the cyclic angular variable $\varphi(\text{mod } 2\pi)$ and defines the family of phase curves $\Lambda^1(I)$ corresponding to a stable electron motion along the circle with the radius $R_0(I)$, the revolution frequency $\omega_0(I)$ and the orbital angular momentum $p_\varphi=I$ (see Appendix 1):

$$\Lambda^1(I)=\{\rho=R_0(I), \varphi=\omega_0(I)\tau, z=0, p_\rho=0, p_\varphi=I, p_z=0\}, \tag{73}$$

where

$$R_0^{2-q}(I)=\frac{cI}{eb}\left(\frac{2-q}{1-q}\right)$$

$\omega_0(I)=eH(R_0)/\varepsilon_0$, $\varepsilon_0=\varepsilon(I)$, and $H(R_0)=b/R_0^q(I)$ is the magnetic field magnitude on the equilibrium orbit (73). The Floquet solutions of the system in variations forming the complex germ $r^3(I)$ have the form

$$a_0(\tau,I)=(0,0,0,0,\omega_0(I),0)^{\mathrm{T}}$$

$$a_1(\tau,I)=\mathrm{e}^{\mathrm{i}\omega_1\tau}(0,0,\mathrm{i}\alpha_1^{-1},0,0,\alpha_1)^{\mathrm{T}} \qquad \omega_1(I)=\omega_0(I)(q)^{1/2} \tag{74}$$

$$a_2(\tau,I)=\mathrm{e}^{\mathrm{i}\omega_2\tau}\left(\mathrm{i}\alpha_2^{-1},0,0,\alpha_2,\frac{\mathrm{i}\alpha_2}{R_0(1-q)^{1/2}},0\right)^T \qquad \omega_2(I)=\omega_0(I)(1-q)^{1/2}$$

where $\alpha_k=(c/\varepsilon_0\omega_k)^{1/2}$, $k=1,2$. According to equation (9) the family of hyperplanes $\tau=\tau(q)=\varphi/\omega_0(I)$ corresponds to the $\Lambda^1(I)$ curve.

Now, let an explict expression for energy spectrum (67) be obtained. Condition (65) leads to quantization of the electron orbital angular momentum: $I=\hbar l(\hbar)$. Here, a set of integer numbers $l(\hbar)$ is defined by the condition

$$\lim_{\hbar\to 0} \hbar l(\hbar)=I_0$$

where the value $I_0$ corresponds to the $\Lambda^1(I_0)$ orbit lying at a given energy level $E_0(I_0)=\lambda^{(+)}|_{\Lambda^1(I_0)}$. Calculate the energy correction $E^{\zeta}_{\zeta}$ due to the interaction of the electron spin with the external field. If no torsion fields are available, the 'polarization' vector $\mathfrak{B}(\tau)$, defined in (56) is given by

$$\mathfrak{B}=-\frac{e}{2\varepsilon_0}\boldsymbol{H}(R_0) \tag{75}$$

where $\boldsymbol{H}(R_0)=(0,0,-H(R_0))$ and is constant. Whence, according to (A2.5) one has $E^{\zeta}_{\zeta}=\hbar\omega^{\zeta}_{\zeta}=\hbar\zeta|\mathfrak{B}|=\frac{1}{2}\zeta\hbar\omega_0$, $\zeta=\pm 1$. The expression for the quasi-classical series of the energy levels takes the form

$$\begin{gathered} E_{l,\nu_1,\nu_2,\zeta}(\hbar)=E^{(0)}_l(\hbar)+\frac{\zeta\hbar}{2}\omega_0(\hbar l)+\hbar\sum_{k=1}^{2}\omega_k(\hbar l)(\nu_k+1/2)+O(\hbar^2) \\ \nu_1,\nu_2=0,1,2,\ldots \qquad l=\pm 1,\pm 2,\ldots \qquad \zeta=\pm 1 \end{gathered} \tag{76}$$

where $E^{(0)}_l(\hbar)=(e^2R_0^2(\hbar l)H^2(R_0(\hbar l))+m^2c^4)^{1/2}$ is the leading term of the electron energy. Now, make use of the definition of the cyclic frequency

$$\omega_0(I)=\frac{\partial E_0(I)}{\partial I}.$$

Then, with the accuracy to $O(\hbar^2)$ the following expansion is true:

$$E^{(0)}_{l+\zeta/2}(\hbar)=E^{(0)}_l(\hbar)+\frac{\zeta\hbar}{2}\omega_0(\hbar l)+O(\hbar^2) \qquad \hbar\to 0,$$

from which it follows that

$$E_{l,\nu_1,\nu_2,\zeta}(\hbar)=E^{(0)}_{l+\zeta/2}(\hbar)+\hbar\sum_{k=1}^{2}\omega_k(\hbar l)(\nu_k+1/2)+O(\hbar^2). \tag{77}$$

The value $(l+\zeta/2)\hbar$ is likely to be a total angular electon momentum allowing for its spin.

For the quasi-classical wavefunctions $\mathring{\Psi}_{E_N}$ (60) corresponding to the energy levels (77), it is not difficult to obtain the following expression

$$\begin{aligned} \mathring{\Psi}_{E_{l,\nu_1,\nu_2,\zeta}}(q,\hbar)=&\frac{e^{il\varphi}}{\pi}\frac{(W_1W_2)^{1/2}}{(2\hbar R_0 2^{\nu_1+\nu_2}\nu_1!\,\nu_2!)^{1/2}} \\ &\times\left\{\exp(-W_1^2z^2/2\hbar)H_{\nu_1}\left(W_1\frac{z}{(\hbar)^{1/2}}\right)\exp(-W_2^2(\rho-R_0)^2/2\hbar)\right. \\ &\left.\times H_{\nu_2}\left(W_2\frac{(\rho-R_0)}{(\hbar)^{1/2}}\right)\right\}\Pi_+(\tau)f_\zeta \end{aligned} \tag{78}$$

where

$$W_1=\left(\frac{eH(R_0)}{c}(q)^{1/2}\right)^{1/2} \qquad W_2=\left(\frac{eH(R_0)}{c}(1-q)^{1/2}\right)^{1/2} \qquad R_0=R_0(\hbar l)$$

$H_\nu(\xi)$ are Hermite polynomials, and the constant spinors $f_\xi$, $\zeta=\pm 1$ are defined in (A2.5). It follows from the form of functions (78) that at $\hbar\to 0$ they are localized in the neighbourhood of the equilibirium circle with radius $R_0(I)$ and that they form a complete orthonormal set of functions.

It should be noted that the quasi-classical energy spectrum (77) constructed above coincides with that obtained earlier in [8] by the variable separation method in the harmonic aproximation.

### 4.2. *Electron spectral series in the class of axially symmetric focusing electric fields*

Consider an electron motion in an axially symmetric focusing electric field which, in cylindrical coordinates $(q^a)=(\rho,\varphi,z)$, is defined by the potentials

$$A_\rho=A_\varphi=A_z=0 \qquad A_0=\Phi_0\rho^\mu \tag{79}$$

where $\Phi_0=$ constant, and $\mu$ is the focusing parameter.

In this case the Dirac operator $\hat{H}_D$ permits the symmetry operator $\hat{p}_z=-i\hbar\partial_z$, therefore, at the very beginning it is reasonable to separate the variable $z$ and look for the quasi-classical stationary states $\Psi_E$ in the form

$$\Psi_E(q,\hbar)=e^{ik_z z}\tilde{\Psi}_E(\rho,\varphi,\hbar). \tag{80}$$

The classical Hamiltonian function corresponds to the function $\tilde{\Psi}_E$

$$\lambda^{(+)}(p,q)=c\varepsilon(p,q)+e\Phi_0\rho^\mu \tag{81}$$

where $\varepsilon(p,q)=(p_\rho^2+\rho^{-2}p_\varphi^2+k_z^2+m^2c^2)^{1/2}$. In the phase space possessing the coordinates $(p_\rho,p_\varphi,\rho,\varphi)$ the Hamiltonian system with the Hamiltonian (81) defines a family of closed phase curves

$$\Lambda^1(I)=\{\rho=R_0(I),\ \varphi=\omega_0(I)\tau,\ p_\rho=0,\ p_\varphi=I\} \tag{82}$$

describing the electron motion along the equilibrium circle with radius $R_0(I)$, where $R_0^{\mu+2}(I)=cI^2/(e\varepsilon_0\Phi_0\mu)$, $\varepsilon_0=\varepsilon(I)$ (hereafter, the condition $e\Phi_0\mu>0$ should be considered to be fulfilled), with the revolution frequency $\omega_0(I)=v_\perp(I)\,\mathrm{sign}\,I/R_0(I)$, where $v_\perp(I)=c|I|/\varepsilon_0R_0(I)$. It should be noted that the state $\tilde{\Psi}_E$ corresponds to the electron motion in the 'reduced' configuration space along a circle with radius $R_0(I)$, while the motion along a spiral with the same radius and the velocity along the axis $z$ equal to $\dot{z}=ck_z/\varepsilon_0$ corresponds to the state $\Psi_E$ in the 'total' configuration space $(\rho,\varphi,z)$.

The electron energy at the equilibrium revolution orbit is equal to

$$E_0(I)=\lambda^{(+)}|_{\Lambda^1(I_0)}=e\Phi_0R_0^\mu(\mathrm{I})+c\varepsilon_0. \tag{83}$$

The Floquet solution of the system in variations which is skew-orthogonal to the vector $a_0(\tau,I)=(0,0,0,\omega_0(I))$ is defined as follows

$$\begin{aligned}\alpha(\tau,I)&=e^{i\omega\tau}(\alpha,0,-i\alpha^{-1},-W_{p_\varphi\rho}/\alpha\omega)^T\\ \omega(I)&=|\omega_0(I)|(\mu+2-v_\perp^2/c^2)^{1/2}\end{aligned} \tag{84}$$

where

$$W_{p_\varphi\rho}=\frac{\omega_0}{R_0}(-2+v_\perp^2/c^2), \qquad \alpha=(\varepsilon_0\omega_0(I)/c)^{1/2}.$$

A condition of existence of the complex germ $r^n(I)$ formed by the vectors $a_0(\tau,I)$ and $a(\tau,I)$ results from (84): $\mu+2-u_\perp^2/c^2>0$.

Now we shall turn to the construction of the quasi-classical spectrum of the energy levels $E_N(\hbar)$ corresponding at $\hbar\to 0$ to an electron motion along the equilibrium orbit $\Lambda^1(I_0)$ with the energy $E_0=E_0(I)$. The condition of the angular momentum quanitization $I=\hbar l(\hbar)$, where $\hbar l(\hbar)\to I_0$ at $\hbar\to 0$ results from equation (65). Then, according to (9) and (56) one has

$$\mathcal{B}(\tau)=\frac{1}{2}\frac{v_\perp^2}{c^2(1+\gamma^{-1})}\left(\frac{ck_z}{\varepsilon_0 R_0}\sin\omega_0\tau,\ -\frac{ck_z}{\varepsilon_0 R_0}\cos\omega_0\tau,\ \omega_0\right) \tag{85}$$

where $\tau=\varphi/\omega_0(I)$. Thus, we turn to case 2 considered in Appendix 2. Making use of (A2.10) it is not difficult to get the expression for the spin-orbital interaction energy $E_\zeta^s$:

$$E_\zeta^s=\hbar\omega_\zeta^s=\zeta\left(\frac{\hbar\omega_0}{2}+\frac{\hbar|\omega_0|}{2}(1-v_\perp^2/c^2)^{1/2}\right) \qquad \zeta=\pm 1. \tag{86}$$

Finally, for the energy levels of an electron we shall obtain

$$\begin{aligned}E_{l,\nu,\zeta}(\hbar)&=E^{(0)}_{l+\zeta/2}(\hbar)+\hbar|\omega_0(I)|(2+\mu-v_\perp^2(I)/c^2)^{1/2}\\&\times\left\{\nu+\frac{1}{2}+\frac{\zeta}{2}\frac{(1-v_\perp^2(I)/c^2)^{1/2}}{(2+\mu-v_\perp^2(I)/c^2)^{1/2}}\right\}\Bigg|_{I=\hbar l(\hbar)}+O(\hbar^2) \qquad \nu=0,1,2,\ldots\\&l=\pm1,\pm2,\ldots \qquad \zeta=\pm1.\end{aligned} \tag{87}$$

In partaicular, it follows from (87) that there is no explicit dependence of the electron energy on its spin for the electric field of the 'Coulomb' type ($\mu=-1$). In this case a renumbering of the energy evels takes place: $E_{l,\nu,\zeta}(\hbar)\to E_{l,\nu'}(\hbar)$, where $\nu'=\nu+(1+\zeta)/2$.

The quasi-classical set of the orthonormal functions corresponding to the energy levels (87) is represented in the form

$$\Psi_{E_{l,\nu,\zeta}}(q,\hbar)=\frac{e^{ik_z z}\,e^{il\varphi}}{2\pi^{5/2}\hbar^{1/4}}\frac{(\alpha)^{1/2}}{(R_0 2^\nu \nu!)^{1/2}}\left\{\exp(-\alpha^2(\rho-R_0)^2/2\hbar)H_\nu\left(\alpha\frac{(\rho-R_0)}{(\hbar)^{1/2}}\right)\right\}\Pi_+(\tau)f_\zeta \tag{88}$$

where $R_0=R_0(\hbar l)$. The expressions for the spinors $f_\zeta$, $\zeta=\pm1$, are given in (A2.10). The functions (88) are localized at $\hbar\to 0$ in a small neighbourhood of the circle with radius $R_0=R_0(I_0)$.

### *4.3. Spectral series of the relativistic electron moving along equatorial orbits in the Coulomb field*

In the Coulomb field with the potential $A_0=Ze_0/\rho$ the classical Hamiltonian function is as follows

$$\lambda^{(+)}(p,q) = -\frac{Ze_0^2}{\rho} + c(p_\rho^2 + \rho^{-2}p_\theta^2 + \rho^{-2}\sin^{-2}\theta p_\varphi^2 + m^2c^2)^{1/2}$$
$$= -\frac{Ze_0^2}{\rho} + c\varepsilon(p,q) \tag{89}$$

where $(q^\alpha) = (\rho, \theta, \varphi)$ are the spherical coordinates, and $e = -e_0$ is an electron charge. A family of closed phase curves $\Lambda^1(I)$ corresponds to the extreme points of the function $\tilde{\lambda}_I^{(+)}$ see Appendix 1):

$$\Lambda^1(I) = \{p_\rho = 0, p_\theta = 0, p_\varphi = I, \rho = R_0(I), \theta = \pi/2, \varphi = \omega_0(I)\tau\} \tag{90}$$

which describe the stationary revolutions of an electron along the equatorial orbits with radius $R_0(I) = I^2/[Ze_0^2 m\gamma(I)]$ and frequency $\omega_0(I) = v_\perp(I)/R_0(I)$, where $\gamma = (1 - v_\perp^2/c^2)^{1/2}$ and $v_\perp(I) = Ze_0^2/I$. The electron energy at the equilibrium orbit is equal to $E_0(I) = mc^2\gamma^{-1}(I)$. The complex germ is formed by the vectors

$$a_0(\tau, I) = (0,0,0,0,\omega_0(I),0)^{\mathrm{T}}$$
$$a_1(\tau, I) = \mathrm{e}^{\mathrm{i}\omega_0\tau}(0, \mathrm{i}\alpha_1^{-1}, 0, 0, \alpha_1, 0)^{\mathrm{T}} \tag{91}$$
$$a_2(\tau, I) = \mathrm{e}^{\mathrm{i}\omega_2\tau}\left(\mathrm{i}\alpha_2^{-1}, 0, 0, \alpha_2, 0, -\frac{\mathrm{i}\alpha_2}{\omega_2}W_{\rho p_\varphi}\right)^{\mathrm{T}} \qquad \omega_2(I) = \omega_0(I)\gamma^{-1}(I)$$

where $\alpha_1 = I^{-1/2}$, $\alpha_2 = (c/(\varepsilon_0\omega_2))^{1/2}$, $\varepsilon_0 = \varepsilon(I)$, $W_{\rho p_\varphi} = (\omega_0/R_0)(-2 + v_\perp^2/c^2)$. The function $\tau(q)$ defined by (9) has the form $\tau = \varphi/\omega_0(I)$. The 'polarization' vector $\mathscr{B}$ (56) is directed along the axis $0z$ and is given by

$$\mathscr{B}(\tau) = \frac{Ze_0^2\omega_0}{2c(1+\gamma^{-1})\varepsilon_0 R_0} e_z. \tag{92}$$

From formula (A2.5) one finds the spin correction

$$E_\zeta^s = \hbar\omega_\zeta^s = \frac{\hbar\zeta}{2}\omega_0(1-\gamma^{-1}).$$

Then the quasi-classical energy spectrum becomes

$$E_{l,\nu_1,\nu_2,\zeta}(\hbar) = \left[E_0(I) + \hbar\omega_0(I)\left(\nu_1 + \frac{1+\zeta}{2}\right) + \hbar\omega_0(I)\gamma^{-1}(I)\left(\nu_2 + \frac{1-\zeta}{2}\right)\right]_{I=\hbar l(\hbar)} + O(\hbar^2)$$
$$\nu_1, \nu_2 = 0, 1, 2, \ldots \qquad l = 1, 2, \ldots \qquad \zeta = \pm 1. \tag{93}$$

For the corresponding leading terms of the quasi-classical eigenstates (60) one gets

$$\mathring{\Psi}_{E_{l,\nu_1,\nu_2,\zeta}}(q,\hbar) = \frac{\mathrm{e}^{\mathrm{i}l\varphi}}{\pi R_0}\frac{(W_1W_2)^{1/2}}{(2\hbar 2^{\nu_1+\nu_2}\nu_1!\,\nu_2!)^{1/2}}$$
$$\times\left\{\exp\left(-W_1^2\frac{(\theta-\pi/2)^2}{2\hbar}\right)H_{\nu_1}\left(W_1\frac{(\theta-\pi/2)}{\hbar^{1/2}}\right)\exp\left(-W_2^2\frac{(\rho-R_0)^2}{2\hbar}\right)\right.$$
$$\left.\times H_{\nu_2}\left(W_2\frac{(\rho-R_0)}{\hbar^{1/2}}\right)\right\}\Pi_+(\tau)f_\zeta. \tag{94}$$

Here, $W_k = \alpha_k^{-1}$, $k = 1, 2$, $R_0 = R_0(\hbar l)$, $H_\nu(\xi)$ are Hermite polynomials, and $f_\zeta$ are the two-component spinors defined by (A2.5).

Within the non-relativistic limit $v_\perp \ll c$ formula (93) gives a quasi-classical series of eigenvalues of the Schrödinger operator obtained in [5]:

$$E_{l,\nu_1,\nu_2}(\hbar) = -mc^2 \frac{\alpha^2 Z^2}{2}\left(\frac{1}{l^2(\hbar)} - \frac{2}{l^3(\hbar)}(\nu_1+\nu_2+1)\right) + O(\hbar^2) \tag{95}$$

where $\alpha = e_0^2/\hbar c$ is the fine structure constant. A correspondence between quantum numbers of the exact and approximated spectrum was also established there: $n_l = l$, $n_r = \nu_1 + \nu_2$, $n_l \approx 1/\hbar$, where $n_r$ and $n_l$ are the radial and orbital quantum numbers, respectively. It would be interesting to compare (93) with the exact energy spectrum of a hydrogen-like atom obtained according to the Dirac theory. In this case one has $n_l = l + \nu_1$, $n_r = \nu_2 + \frac{1}{2}(1-\zeta)$, $n_l \approx 1/\hbar$, $\hbar \to 0$.

It should be noted in conclusion that it is not difficult to generalize the results obtained in this section for the case when the external torsion fields are non-zero (if any), and in this way to estimate their possible influence on the quasi-classical energy spectrum. As an example consider a hydrogen atom in the external torsion field of the type

$$S_0 = S_0(\rho) \qquad S = (-S_\perp(\rho)\sin\varphi, S_\perp(\rho)\cos\varphi, S_3(\rho)) \tag{96}$$

and calculate in this case the spin–orbit coupling energy of an electron $E_\zeta^s = \hbar\omega_\zeta^s$ moving along the equatorial orbit $l_I = (\rho = R_0(I), \theta = \pi/2, \varphi = \omega_0(I)\tau)$. In this case the polarization vector $\mathfrak{B}(\tau)$ (56) is given by $\mathfrak{B}(\tau) = (-\mathfrak{B}_\perp \sin\omega_0\tau, \mathfrak{B}_\perp\cos\omega_0\tau, \mathfrak{B}_3)$ where

$$\begin{aligned}\mathfrak{B}_\perp &= S_\perp(R_0) - \frac{v_\perp}{c} S_0(R_0)\\ \mathfrak{B}_3 &= \frac{\omega_0}{2} + (1 - v_\perp^2/c^2)^{1/2}\left(S_3(R_0) - \frac{\omega_0}{2}\right)\end{aligned} \tag{97}$$

from which, according to formula (A2.10), one obtains

$$\begin{aligned}E_\zeta^s(\hbar l(\hbar)) &= \hbar\omega_\zeta^s(\hbar l(\hbar))\\ &= \zeta\hbar\left(\frac{\omega_0(I)}{2} + \left(\mathfrak{B}_\perp^2 + \left(\mathfrak{B}_3(I) - \frac{\omega_0(I)}{2}\right)^2\right)^{1/2}\right)_{I=\hbar l(\hbar)}.\end{aligned} \tag{98}$$

## 5. Conclusions

The construction of the quasi-classical spectral series discussed here is substantiated by an additional assumption that there is no focal points where $|\dot{Q}(\tau)| = 0$ on the closed phase curve $\Lambda^1(E_0)$. This assumption may not hold true for closed curves of multidimensional non-integrable systems [12]. In this case quantization of closed trajectories by the complex germ method requires that a number of new auxiliary constructions [3, 6, 11] should be used.

In the formulas of sections 2 and 3 it was implicitly supposed that equation (9) defining the hyperplane family $\tau = \tau(q)$ is solvably smoothly and in a single-valued way in the small neighbourhood $U_\delta(E_0)$ throughout the closed curve $l_{E_0}$. If this is not the case, then the curve $l_{E_0}$ should be covered by the neighbourhood $U_\delta^j(E_0), j=1,\ldots,m$, in each of which equation (9) permits a smooth solution $\tau^j=\tau^j(q)$, and a set of geometrical objects $[\Lambda_j^1(E_0), r_j^n(E_0)]$ should be constructed and united in a special way (for details see [4]).

Approaching the quasi-classical quantization of closed orbits by the complex germ method it is interesting to consider the following problems:

- construction of the quasi-classical spectral series for the case when the closed orbits $l_{E_0}$ (a projection of $\Lambda^1(E_0)$ onto the configuration space) possesses self-intersection points;
- solution of the spectral problem in the region of chaotic behaviour of the corresponding classical system [12, 13];
- relationship between the quantum Berry's phase and the construction of Maslov's canonical operator with complex phase in the case of matrix wave equations [14–16].

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**Appendix 1. Construction of the family $[\Lambda^1(I), r^n(I)]$ for classical systems with one cyclic variable**

Let a case often encountered in applications be considered, when the Hamiltonian function (1) permits a cyclic angular variable $\varphi(\mathrm{mod}\,2\pi)$. In this case it is always possible to separate a spacial family of closed phase curves $\Lambda^1(I)$ corresponding to the motion of a classical particle along an equilibrium circle. Let their construction be briefly described (see also [51]).

Denote the conjugated momenta to the variable $\varphi$ as $I$. Fix the numerical interval $\Omega_I \ni I$ and let the condition

$$\frac{\partial H}{\partial I}(p, I, q) \neq 0 \tag{A1.1}$$

be fulfilled. Introduce a one-dimensioal in $I$ family of the Hamiltonian functions on the reduced phase space $\mathbb{R}_p^{n-1} \times \mathbb{R}_q^{n-1}$

$$\tilde{H}_I(p, q) = H(p, I, q) \qquad I \in \Omega_I. \tag{A1.2}$$

Denote the extreme point of the function $\tilde{H}_I(p, q)$ as $\tilde{r}_0(I) = (p_0(I), q_0(I))$:

$$\nabla_p \tilde{H}_I|_{\tilde{r}_0(I)} = \nabla_q \tilde{H}_I|_{\tilde{r}_0(I)} = 0. \tag{A1.3}$$

Then it is not difficult to see that at each fixed value of the parameter $I \in \Omega_I$ the curve

$$\Lambda^1(I) = \{p_0(I), I, q_0(I), \varphi = \omega_0(\mathrm{I})\tau\} \tag{A1.4}$$

where

$$\omega_0(I)=\frac{\partial H}{\partial I}(p_0(I), I, q_0(I))$$

is a closed trajectory of the initial Hamiltonian system (2) lying at the energy level

$$E_0(I_0)=\lambda^{(+)}|_{\Lambda^1(I_0)}.$$

In the configuration space the motion along a closed equilibrium circle with the period $T(I)=2\pi/\omega_0(I)$ corresponds to the trajectory $\Lambda^1(I)$.

Construction of the complex germ $r^n(I)$ on the family $\Lambda^1(I)$ is essentially simplified, since in this case matrix (5)

$$H_{\mathrm{VAR}}|_{\Lambda^1(I_0)}=W(I)$$

is constant and the procedure of constructing the Floquet solutions satisifying the conditions of the 'germ' existence (8) is reduced to solution of the spectral problem for the matrix $W(I)$:

$$W(I)f_k=\mathrm{i}\omega_k(I)f_k \qquad \mathrm{Im}\,\omega_k=0. \tag{A1.5}$$

Then, the desired solutions of the Floquet system (4) with the matrix $W(I)$ are as follows

$$a_k(\tau, I)=\exp(\mathrm{i}\omega_k(I)\tau)f_k. \tag{A1.6}$$

## Appendix 2

Here, the formulas for the Floquet solutions of the system (58) used in constructing the spectral series of the Dirac operator in Section 4 will be presented.

*Proposition.* All the multiplicators of Eq. (58) are equal to unity in magnitude.

*Proof.* Let $v_\xi$ be a certain Floquet solution of (58) with the multiplicator $\lambda_\xi$, then

$$v_\xi(\tau+T)=\lambda_\xi v_\xi(\tau). \tag{A2.1}$$

Whence it follows

$$\overset{+}{v}_\xi(\tau)+T)v_\xi(\tau+T)=|\lambda_\xi|^2\overset{+}{v}_\xi(\tau)u_\xi(\tau). \tag{A2.2}$$

It should be noted further that for any two solutions of (58) $v_1(\tau)$ and $v_2(\tau)$ the product $\overset{+}{v}_1(\tau)v_2(\tau)$ is conserved, from which it follows that

$$\overset{+}{v}_\xi(\tau+T)v_\xi(\tau+T)=\overset{+}{v}_\xi(\tau)v_\xi(T). \tag{A2.3}$$

Comparing (A2.2) and (A2.3) we shall obtain the proof.

Now consider two particular cases when the problem of the existence of the Floquet solutions satisfying conditions (62), (63) is solved in a rather simple way.

*Case 1.* $\mathcal{B}(\tau)=\mathcal{B}=$constant. In this case the monodromy matrix of system (58) is $G(T)=\exp(-\mathrm{i}(\sigma\mathcal{B})T)$ and the problem of constructing the multiplicator $\lambda_\xi=\exp(-\mathrm{i}\omega_\xi^\xi T))$ is reduced to solution of the equation

$$(\sigma\mathcal{B})f_\xi=\omega_\xi^\xi f_\xi. \tag{A2.4}$$

Introduce the unit vector $\boldsymbol{n}=\mathcal{B}/|\mathcal{B}|=(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$. Then (A2.4) permits a general solution of the type

$$\omega_\zeta^\xi=\zeta|\mathcal{B}| \qquad \zeta=\pm 1$$

$$f_\zeta=\frac{\mathrm{e}^{\mathrm{i}\alpha}}{(2)^{1/2}}\begin{pmatrix}\zeta(1+\zeta\cos\theta)^{1/2}\,\mathrm{e}^{-\mathrm{i}\varphi/2}\\ (1-\zeta\cos\theta)^{1/2}\,\mathrm{e}^{\mathrm{i}\varphi/2}\end{pmatrix} \qquad \alpha=\text{constant} \tag{A2.5}$$

where the constant spinors $f_\zeta$ form a complete and orthonormal set

$$\overset{+}{f}_{\zeta'}f_\zeta=\delta_{\zeta'\zeta} \qquad \sum_\zeta f_\zeta\overset{+}{f}^{\zeta}=1. \tag{A2.6}$$

Finally, we obtain the following set of Floquet solutions

$$v_\zeta(\tau)=\exp(-\mathrm{i}\zeta|\mathcal{B}|\tau)f_\zeta \qquad \zeta=\pm 1. \tag{A2.7}$$

*Case 2.* $\mathcal{B}(\tau)=(-\mathcal{B}_\perp\sin\omega_0\tau,\ \mathcal{B}_\perp\cos\omega_0\tau,\ \mathcal{B}_3)$, where $\omega_0=2\pi/T(E_0)$, $\mathcal{B}_\perp$, $\mathcal{B}_3=$ constant. In this case by means of a unitary transformation $v(\tau)=S\tilde{v}(\tau)$, $S=\exp(-\mathrm{i}\omega_0\tau\sigma_3/2)$ equation (58) is reduced to

$$\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\tau}+\langle\boldsymbol{\sigma},\tilde{\mathcal{B}}|\rangle\right)\tilde{v}(\tau)=0 \tag{A2.8}$$

where $\tilde{\mathcal{B}}=(0,\mathcal{B}_\perp,\mathcal{B}_3-\omega_0/2)$ and, therefore, to case 1 considered above. As a result, one can get the following set of Floquet solutions

$$v_\zeta(\tau)=\exp(-\mathrm{i}(\sigma_3\omega_0/2+\zeta|\mathcal{B}|)\tau)\tilde{f}_\zeta \tag{A2.9}$$

with the characteristic indices

$$\omega_\zeta^\xi=\zeta\left(\frac{\omega_0}{2}+|\mathcal{B}|\right) \qquad z=-\pm 1. \tag{A2.10}$$

Here, the spinors $\tilde{f}_\zeta$ are defined by the unit vector $\boldsymbol{n}=\mathcal{B}/|\mathcal{B}|$ according to formula (A2.5).

## Appendix 3

Let the Hamiltonian function $\lambda^{(+)}(p,q)$ permit a smooth family of $T(E)$-periodic closed curves $\Lambda^1(E)=\{p=P(\tau,E),\ q=Q(\tau,E)\}$ such that

$$\lambda^{(+)}|_{\Lambda^1(E)}=E.$$

It may be shown that if $E=E_0+\hbar E_1+O(\hbar^2)$ then the quantization condition (70) and (65) are equivalent with accuracy to $O(\hbar^2)$.

As follows from the comparison of expressions (70) and (64), to prove the latter statement it is sufficient to be convinced that the following relation

$$\oint_{\Lambda^1(E)}\langle P(\tau,E),\mathrm{d}Q(\tau,E)\rangle=\oint_{\Lambda^1(E_0)}\langle P(\tau,E_0),\mathrm{d}Q(\tau,E_0)\rangle+\hbar T(E_0)E_1+O(\hbar^2). \tag{A3.1}$$

Indeed, in view of the smooth dependence of the family $\Lambda^1(E)$ on the parameter $E$, the integral

$$W(E)=\oint_{\Lambda^1(E)}\langle P(\tau,E),\mathrm{d}Q(\tau,E)\rangle$$

permits the expansion

$$W(E)=W(E_0)+\left(\frac{\partial W}{\partial E}\right)\bigg|_{E_0}\hbar E_1+O(\hbar^2).$$

It is easy to see that

$$\frac{\partial W}{\partial E}(E)=T(E)+\langle P(T(E),E),\dot{Q}(T(E),E)\rangle T'(E)+\left\langle P(\tau,E),\frac{\partial Q}{\partial E}(\tau,E)\right\rangle\bigg|_0^{T(E)}. \tag{A3.2}$$

For the $T(E)$-periodic function $Q(\tau,E)$ make use of the Fourier series expansion

$$Q(\tau,E)=\sum_{n=-\infty}^{\infty}\exp\left(\frac{2\pi in\tau}{T(E)}\right)C_n(E).$$

Whence it follows directly

$$\frac{\partial Q}{\partial E}(\tau,E)=-\frac{T'(E)}{T(E)}\tau\frac{\partial Q}{\partial\tau}(\tau,E)+\sum_{n=-\infty}^{\infty}\exp\left(\frac{2\pi in\tau}{T(E)}\right)\frac{\partial C_n}{\partial E}(E). \tag{A3.3}$$

Substituting (A3.3) into (A3.2) and making use of the $T(E)$-periodicity condition of the function $P(\tau,E)$ one gets

$$\frac{\partial E}{\partial E}(E)=T(E). \tag{A3.4}$$

As a result, the expression $W(E)$ takes the form $W(E)=W(E_0)+T(E_0)\hbar E_1+O(\hbar^2)$. This is the proof of the required statement.

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